$\mathbb{Z}_2$ topological phase in quantum antiferromagnets

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RVB spin liquid

4 spins on a square:
Groundstate is exactly
\[ \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ \end{array} \right) \]

\[ = \frac{1}{\sqrt{2}} (| \uparrow \downarrow \rangle - | \downarrow \uparrow \rangle) \]
singlet pair
a.k.a. “valence bond”

So, the groundstate on the square lattice may be the RVB state:

\[
\sum \text{all the valence-bond patterns}
\]

?
Is it true?

The RVB state was expected to be disordered state without any order – “RVB spin liquid”

However, contrary to the initial motivation, the groundstate of the square-lattice Heisenberg model turned out to have Neel order

Frustration might favor the RVB spin liquid. However, no case was established in Heisenberg model on various frustrated lattice (triangular etc.)
S=1/2 Kagome Lattice HAF

prototypical frustrated magnet

Various proposals
Magnetic order?
Gapless? Gapped?
Valence-Bond Crystal?
Spin liquid?
Spin-Peierls and spin-liquid phases of Kagomé quantum antiferromagnets

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Physics of low-energy singlet states of the Kagome lattice quantum Heisenberg antiferromagnet

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FIG. 1. Unit cells with the maximum fraction of perfect hexagons where the thick lines denote dimers. (a) 18-site unit cell with perfect hexagons that form a oblique lattice. (b) 36-site unit cell with perfect hexagons that form a honeycomb lattice.
Frustrated Antiferromagnets with Entanglement Renormalization: Ground State of the Spin-$\frac{1}{2}$ Heisenberg Model on a Kagome Lattice

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Multiscale Entanglement Renormalization Ansatz (MERA)
DMRG for cylinder up to circumference of 12
The ground state that we find has only short-range correlations, with a nonzero energy gap for any excitations, including spin-singlet excitations. We have tested the response of this ground state to many sorts of perturbations that would select out ordered states, if they exist, without detecting any signs of any ordering. Thus, we conclude that the ground state is a gapped spin liquid.
Resonating–Valence–Bond (RVB) state: disordered ground state without Neel order?

History of RVB spin liquid phase

Square lattice Heisenberg: no RVB (Neel order)
[Various authors, - 1989]

Square lattice Quantum Dimer Model:
critical pt. between 2 VBC phases
[Rokhsar-Kivelson 1988]

Triangular lattice QDM:
umerical evidence for gapped RVB
[Moessner-Sondhi 2001]
History of RVB spin liquid phase II

Exactly solvable models of gapped RVB phase = “$Z_2$ topological phase”

“Toric code” [A. Kitaev (1997)]

Kagome lattice QDM [Misguich-Serban-Pasquier (2002)]
RVB and “loop gas”

Reference state = Valence Bond Crystal (VBC)

Reference

“new” state
RVB and “loop gas”

Reference state = Valence Bond Crystal (VBC)

“resonance loop”

Reference

“new” state
$\mathbb{Z}_2$ topological phase

is given by a quantum superposition of configurations with closed loops

Proliferation (condensation) of loops

$\Leftrightarrow \mathbb{Z}_2$ topological phase
Non-condensation of loops:
VBC order is preserved (conventional order)

Condensation (proliferation) of loops:
Destruction of VBC order, without any conventional (magnetic, etc.) orders
= RVB phase
Problem:
There are now many (artificial) models which are known to belong to topological phases

How we can realize these phases, such as “spin liquid phase” in experiment, or in realistic models? ⇒

\[ S=1/2 \text{ HAF on kagome lattice?} \]

Even if it is the case, how can we verify the existence of the topological phase, experimentally or numerically?
Identifying topological phases

Experimental/numerical verification of “spin liquid” usually starts with showing absence of any long-range order (magnetic, dimer, etc.) = “negative” evidence

“positive” experimental evidences may be obtained from excitations with fractional charge and/or statistics, etc. but more complicated

It has been also difficult to obtain positive evidences in numerics
Fractionalization in RVB liquid

Removing one dimer (i.e. doping a “hole”) can create two “half-dimer” hole excitations (spinons = fractionalized particle)
Fractionalization in RVB liquid

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Fractionalization in RVB liquid

Removing one dimer (i.e. doping a “hole”) can create two “half-dimer” hole excitations (spinons = fractionalized particle)
Fractionalized excitations in the spin-liquid state of a kagome-lattice antiferromagnet

Tian-Heng Han, Joel S. Helton, Shaoyan Chu, Daniel G. Nocera, Jose A. Rodriguez-Rivera, Collin Broholm & Young S. Lee
Recent neutron scattering expt. on herbertsmithite (T. H. Han, et al.)

- fractionalized spinon excitation

  gapless excitation?
  (inconsistent with the $\mathbb{Z}_2$ spin liquid, maybe the gap is too small....)
Topological Entanglement Entropy

Kitaev-Preskill, Levin-Wen 2006

\[ \rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi| \]

\[ S_E = -\text{Tr}[\rho_A \log \rho_A] \]

\[ S_E \sim \alpha \mathcal{L} - \gamma_{\text{topo}} \]

\( \mathcal{L} \): Boundary length

“area law” term

(non-universal coefficient \( \alpha \))

universal TEE
TEE in Z2 topological phase

Each boundary link may be crossed ($q_i=1$) or not ($q_i=0$). $N_q = 2^\mathcal{L}$?

In fact, the number of string crossings at the boundary must be even, since the strings form closed loops.

$S_E = \log N_q = \mathcal{L} \log 2 - \log 2$

“area law” TEE
TEE is a ground-state property

No need to look at excitations!

Measuring TEE in experiments is unrealistic, but for numerics it looks a convenient quantity characterizing topological phases

However, in numerics, we need to extract the subleading constant term, which is TEE

\[ S_E \sim \alpha \Sigma - \gamma_{\text{topo}} \]
How to extract?

Kitaev-Preskill

\[ S_{\text{topo}} = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC}. \]

Cancellation of “area law” term → extraction of universal constant (TEE)

However, in practical applications, large finite-size effect due to curvature can be problematic

cf.) curvature expansion \[\text{[Grover-Turner-Vishwanath 2011]}\]
Simpler approach

Measure EE for various boundary lengths, then “extrapolate” to zero length
“Smooth” boundary gives less finite-size effects

Problem: generally, there is an ambiguity in defining the boundary length - which leads to ambiguity in estimate of TEE

Solution:

\[ S_E \sim \alpha \mathcal{L} - \gamma_{\text{topo}} \]
Quantum Dimer Model on triangular lattice (Z\textsubscript{2} topological phase) extrapolation works quite well

[Furukawa-Misguich 2007]
$\mathbb{Z}_2$ topological phase on cylinder

Strings can form winding loops along the circumference

Winding number is conserved modulo $2$

$\Rightarrow$ doubly degenerate GS

$|\xi_{0,1}\rangle$ winding number $= 0,1 \ (\text{modulo } 2)$
EE and winding number

Consider one of the “pure” GS $|\xi_{0,1}\rangle$

Crossing # at the cut = winding # (mod 2)

$|\xi_1\rangle = \frac{1}{\sqrt{2N_q}} \sum_{\{q_i\}} \left( |\Psi^A_{\{q_i\},0}\rangle |\Psi^B_{\{q_i\},1}\rangle + |\Psi^A_{\{q_i\},1}\rangle |\Psi^B_{\{q_i\},0}\rangle \right)$

crossing # at the cut (mod 2) within A or B

$S_E = \mathcal{L} \log 2$

“Topological EE” vanishes, owing to the extra entanglement
TEE depends on the groundstate, in the presence of topological degeneracy!

Dong-Fradkin-Leigh-Nowling 2008
Zhang-Grover-Turner-MO-Vishwanath 2011

Useful information on quasiparticle statistics can be obtained from this dependence
Minimal Entropy States (MES) 

Zhang-Grover-Turner-MO-Vishwanath 2011

Among the topologically degenerate groundstates, there are “minimal entropy states” with maximal TEE

\[ S_E = \mathcal{L} \log 2 - \left[ \log 2 - S_{cl}(\{\tilde{p}_0, \tilde{p}_1\}) \right] \]

In the MES, the universal TEE of \( \log 2 \) (for \( \mathbb{Z}_2 \) topological phase) is recovered
DMRG for cylinder up to circumference of 12
Positive evidence for TP?

Problems:

1) **No clear evidence of topological degeneracy** on cylinder, from DMRG
2) TEE may be used as an evidence of Z2 topological phase - but what to do about the groundstate dependence of TEE??

These may be solved naturally

DMRG gives $\gamma_{\text{topo}} = \log 2$

Kagome w/ NNN exchange $J_2$  

[Jiang-Wang-Balents 2012]

Figure 3: The entanglement entropy $S(L_y)$ of the kagomé $J_1$-$J_2$ model in Eq.(2), with $L_y = 4 \sim 12$ at $L_x = \infty$. By fitting $S(L_y) = aL_y - \gamma$, we get $\gamma = 0.698(8)$ at $J_2 = 0.10$, and $\gamma = 0.694(6)$ at $J_2 = 0.15$. Inset: kagomé lattice with $L_x = 12$ and $L_y = 8$. 

Why?

Why the “ideal” value of TEE was obtained, without selecting the MES intentionally?

Answer: DMRG naturally favors MES, and groundstates other than MES requires exponentially large number of states kept “$m$” and cannot be realistically seen in larger systems [Jiang-Wang-Balents 2012]

This also explains the lack of evidences for topological degeneracy from DMRG
demonstration for toric code in magnetic field

[Jiang-Wang-Balents 2012]
DMRG on “pure” kagome (without J2) also shows TEE consistent with Z2 TP

[Depenbrock-McCulloch-Schollwöck 2012]

FIG. 6. Renyi entropies $S_\alpha$ of infinitely long cylinders for various $\alpha$ versus circumference $c$, extrapolated to $c = 0$. The negative intercept is the topological entanglement entropy $\gamma$. 
“F-theorem” in 2+1D CFT

Entanglement entropy across a circle

\[ S_E \sim \alpha L - \gamma \]

coincides with the “topological EE” for topological phases, but can be also defined for critical points (CFT)

Casini-Huerta-Myers: \( \gamma \) is also the universal part of the free energy of the CFT on 3-sphere

Myers-Sinha: \( \gamma \) decreases monotonically along RG!
Application of $F$-theorem


Free boson (scalar): $\gamma \sim 0.0638$

$O(3)$ Gaussian critical point: $\gamma_{O(3)} < 0.0638 \times 3$

Ising critical point: $\gamma_{\text{Ising}} < 0.0638$

$Z_2$ topological phase: $\gamma = \log(2) = 0.693$

There cannot be an RG flow from $O(3)$ Gaussian or Ising critical point to the $Z_2$ topological phase!

(explains why it is difficult to realize the topological phase?)
Conclusion

“(Gapped) RVB spin liquid phase” had been elusive for long time. But its existence is now well established theoretically, at least in special models. It is “$\mathbb{Z}_2$ topological phase” in modern terminology.

The standard $S=1/2$ Heisenberg AFM on kagome lattice may realize the RVB spin liquid phase - positive numerical evidence from Topological Entanglement Entropy.